Recall from last time:

Theorem (Truncated Perron formula)

Let $c > 0$, $T, X > 2$ and $f \in \mathcal{A}$.

If $c > \sigma_a(f)$ and $c > 1$, then

$$\sum_{n \le X} f(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} L_f(s) X^s \frac{ds}{s} + E_{X,c}(T),$$

where $|E_{X,c}(T)| \ll \frac{X^c}{T} \sum_{n=1}^{\infty} \frac{|f(n)|}{n^c} + \max_{\frac{3}{4}X \le n \le \frac{5}{4}X} |f(n)| \left(\log T + \frac{X \log X}{T}\right)$

Corollary: If $f(n) \ll (\log n)^k$ and $2 \le T \le 2X$,

Set $c = 1 + \frac{1}{\log X}$. Then

$$\sum_{n \le X} f(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} L_f(s) X^s \frac{ds}{s} + O\left(\frac{X (\log X)^{k+1}}{T}\right).$$

Proof of corollary: Note that if $c = 1 + \frac{1}{\log X}$,

then $X^c = e \cdot X \ll X$.

$$\sum_{n=1}^{\infty} \frac{|f(n)|}{n^c} \ll \sum_{n=1}^{\infty} \frac{(\log n)^k}{n^c} \ll \sum_{n=1}^{\infty} \frac{n^{\frac{c-1}{2}}}{n^c}$$

$$= \zeta\left(\frac{1+c}{2}\right) = \frac{1}{\frac{1+c}{2} - 1} + O(1) \ll \log X.$$

✓

Application: For any $\varepsilon > 0$, we have
$$\sum_{n \le x} \mu^2(n) = \frac{1}{\zeta(2)} x + O\left(x^{\frac{3}{5}+\varepsilon}\right).$$

(we will use without proof that for $0 < \sigma \le 1$, $|t| \ge 2$, we have $\zeta(\sigma+it) \ll_\varepsilon (1+|t|)^{\frac{1-\sigma}{2}+\varepsilon}$.
we prove this later in the course).

[compare with Ex 2, Sheet 2, where we show there exists a constant $c > 0$ s.t.
$$\sum_{n \le x} \mu^2(n) = c \cdot x + O(\sqrt{x}).$$
This implies $\sum_{n \le x} \mu^2(n) = \frac{1}{\zeta(2)} x + O(\sqrt{x})$]

Pf: Since $L_{\mu^2}(s) = \frac{\zeta(s)}{\zeta(2s)}$, we have that
$$\sum_{n \le x} \mu^2(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\zeta(s)}{\zeta(2s)} x^s \frac{ds}{s} + O\left(\frac{x \lg x}{T}\right),$$
where $c = 1 + \frac{1}{\lg x}$.

Fix $\varepsilon > 0$. We want to shift line of integration to $\mathrm{Re}(s) = \frac{1}{2} + \varepsilon$ and apply the residue theorem.

$\frac{X^s}{s}$ has no pole in $\{Re(s) > \frac{1}{2}\}$.

$\zeta(s)$ has a simple pole at $s=1$ in $\{Re(s) > \frac{1}{2}\}$

$\frac{1}{\zeta(2s)}$ is holomorphic and uniformly bounded in $\{Re(s) > \frac{1}{2}+\varepsilon\}$

Indeed, since if $Re(s) > \frac{1}{2}+\varepsilon$, then $Re(2s) > 1+2\varepsilon$,

$$\frac{1}{\zeta(2s)} = L_\mu(2s) = \sum_{n \geq 1} \frac{\mu(n)}{n^{2s}}$$

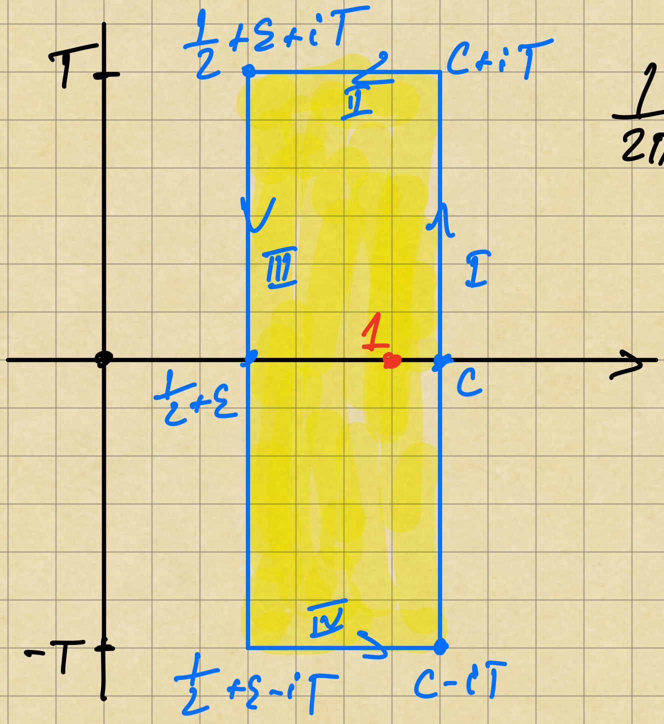
(we are in the area of absolute convergence)

$$\left| \frac{1}{\zeta(2(\sigma+it))} \right| \leq \sum_{n\geq 1} \left| \frac{\mu(n)}{n^{2(\sigma+it)}} \right| \qquad (\mu(n) \in \{0, \pm 1\})$$

$$\leq \sum_{n \geq 1} \frac{1}{n^{2\sigma}} = \zeta(2\sigma) \leq \zeta(1+2\varepsilon) = O_\varepsilon(1).$$

Let $F(s) = \frac{\zeta(s)}{\zeta(2s)} \cdot \frac{X^s}{s}$. We apply residue theorem in the box with corners $c-iT$, $c+iT$, $\frac{1}{2}+\varepsilon+iT$, $\frac{1}{2}+\varepsilon-iT$.

$F(s)$ has only one simple pole at $s=1$ in this box, and $\operatorname{Res}_{s=1} F(s) = \frac{1}{\zeta(2)} \cdot X$.



$$\frac{1}{2\pi i}\left(\int_{c-iT}^{c+iT} F(s)\,ds + \int_{c+iT}^{\frac{1}{2}+\varepsilon+iT} F(s)\,ds + \int_{\frac{1}{2}+\varepsilon+iT}^{\frac{1}{2}+\varepsilon-iT} F(s)\,ds + \int_{\frac{1}{2}+\varepsilon-iT}^{c-iT} F(s)\,ds\right)$$

$$= \operatorname*{Res}_{s=1} F(s) = \frac{1}{\zeta(2)} X .$$

II: $$\left|\int_{c+iT}^{\frac{1}{2}+\varepsilon+iT} \frac{\zeta(s)}{\zeta(2s)}\frac{X^s}{s}\,ds\right| \ll \frac{1}{T}\int_{\frac{1}{2}+\varepsilon}^{c} |\zeta(\sigma+iT)|\, X^\sigma\, d\sigma$$

$$\ll \frac{1}{T}\int_{\frac{1}{2}+\varepsilon}^{c} X^\sigma\, T^{\max\left(\frac{1-\sigma}{2},\,0\right)+\varepsilon}$$

(we used $\zeta(\sigma+it) \ll (1+|t|)^{\frac{1-\sigma}{2}+\varepsilon}$, for $0<\sigma\le 1$)

$$\ll \frac{X^\varepsilon}{T^{1/2}}\int_{\frac{1}{2}+\varepsilon}^{c}\left(\frac{X}{T^{1/2}}\right)^\sigma d\sigma$$

$(c = 1 + \frac{1}{\log x})$.

$$\ll \frac{X^\varepsilon}{T^{1/2}}\cdot\frac{X^c}{T^{c/2}} \ll \frac{X^{1+\varepsilon}}{T}.$$

(similarly for IV)

III: $$\left|\int_{\frac{1}{2}+\varepsilon+iT}^{\frac{1}{2}+\varepsilon-iT} \frac{\zeta(s)}{\zeta(2s)}\frac{X^s}{s}\right| \ll X^{\frac{1}{2}+\varepsilon}\int_0^T \frac{|\zeta(\frac{1}{2}+\varepsilon+it)|}{1+t}\,dt$$

$$\ll X^{\frac{1}{2}+\varepsilon}\int_0^T (1+t)^{-3/4+\varepsilon} \ll X^{\frac{1}{2}+\varepsilon}\cdot T^{1/4}$$

Putting everything together:

$$\sum_{n\le X} \mu^2(n) = \frac{1}{\zeta(2)} X + O\left(X^{\frac{1}{2}+\varepsilon} T^{1/4} + \frac{X^{1+\varepsilon}}{T}\right)$$

Choose $T = X^{2/5}$. $\square$

<u>Proof of Truncated Perron formula</u>:

Denote $M_X := \max_{\frac{3}{4}X \le n \le \frac{5}{4}X} |f(n)|$.

Recall that for $\delta(y) = \begin{cases} 0, & 0<y<1 \\ \frac{1}{2}, & y=1 \\ 1, & y>1 \end{cases}$,

we have that

$$\delta(y) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} y^s \frac{ds}{s} + O\left(\frac{y^c}{T|\log y|}\right).$$

$(y \neq 1)$.

Let $N > X$. Then

$$\sum_{n\le X} f(n) = \sum_{n\le N} f(n)\, \delta\left(\frac{X}{n}\right) + O(M_X)$$

$$\delta\left(\frac{X}{n}\right) = \begin{cases} 1, & n<X \\ 0, & n>X \\ \frac{1}{2}, & n=X \end{cases}$$

$$= \sum_{\substack{n \le N \\ |n-X|>3}} f(n) \left(\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left(\frac{X}{n}\right)^s \frac{ds}{s} + O\left(\frac{(X/n)^c}{T|\log(\frac{X}{n})|} \right) \right)$$

$$+ O(M_X)$$

Note that for $|n-X| \le 3$, we have $\left(\frac{X}{n}\right)^c = O(1)$ (since $\frac{X}{n} = O(1)$) and we assume $c \asymp 1$).

Hence $\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left(\frac{X}{n}\right)^s \frac{ds}{s} \ll \int_0^T \frac{1}{|c+it|} dt \ll \log T.$

Therefore $\sum_{|n-X| \le 3} f(n) \cdot \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left(\frac{X}{n}\right)^s \frac{ds}{s} \ll M_X \cdot \log T.$

We have that

$$\sum_{n \le X} f(n) = \sum_{n \le N} f(n) \cdot \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left(\frac{X}{n}\right)^s \frac{ds}{s}$$

$$+ O\left(M_X \log T + \sum_{\substack{n \le N \\ |n-X|>3}} f(n) \frac{(X/n)^c}{T|\log(\frac{X}{n})|} \right)$$

Note that for $n > \frac{5}{4} \log X$, $|\log(\frac{X}{n})| \ge \log \frac{5}{4} > 0$.

So $\left| \sum_{\substack{n \le N \\ n > 5/4 X}} f(n) \frac{(X/n)^c}{T|\log(\frac{X}{n})|} \right| \le \frac{1}{\log(\frac{5}{4})} \cdot \frac{X^c}{T} \sum_{n \in \mathbb{N}} \frac{|f(n)|}{n^c}.$

Similarly for $n < \frac{3}{4}x$, $\log\left(\frac{x}{n}\right) \geq \log\left(\frac{4}{3}\right) > 0$,

so $\left|\sum_{n<\frac{3}{4}x} f(n) \cdot \frac{(x/n)^c}{T\left|\log\left(\frac{x}{n}\right)\right|}\right| \leq \frac{1}{\log\left(\frac{4}{3}\right)} \cdot \frac{x^c}{T} \sum_{n\in\mathbb{N}} \frac{|f(n)|}{n^c}$.

Finally, for $\frac{3}{4}x \leq n \leq \frac{5}{4}x$, write $n = \lfloor x \rfloor + h$,

so $\left|\log\left(\frac{n}{x}\right)\right| = \left|\log\left(1 + \left(\frac{n-x}{x}\right)\right)\right| \geq \frac{|h|}{2x}$

(we have used that $\log(1+\delta) \geq \frac{|\delta|}{2}$, for $\delta < \frac{1}{3}$.)

Therefore $\sum_{\substack{\frac{3}{4}x \leq n \leq \frac{5}{4}x \\ |n-x| \geq 3}} f(n) \cdot \frac{(x/n)^c}{T\left|\log\left(\frac{x}{n}\right)\right|}$ $\xrightarrow{} O(1)$

$\left(\frac{4}{5}\right)^c \leq \left(\frac{x}{n}\right)^c \leq \left(\frac{4}{3}\right)^c$

$\ll \frac{M_x}{T} \sum_{2 \leq h \leq \frac{x}{4}} \frac{1}{h} \ll \frac{M_x \log x}{T}$.

Hence we showed $\sum_{n \leq x} f(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left(\sum_{n=1}^{N} \frac{f(n)}{n^s}\right) \cdot \frac{x^s}{s} ds$

$+ O(E_{x,c}(T))$.

But the Dirichlet series $\sum_{n=1}^{N} \frac{f(n)}{n^s}$

converges uniformly absolutely on $\mathrm{Re}(s) \geq c$ (since $c > \sigma_a(f)$), so let $N \to \infty$ gives the result.

$\boxtimes$

Fourier transform

Definition: Let G an abelian topological group. $\hat{G}$ is the set of continuous homomorphisms

$$\chi : G \to S^1.$$

Examples:

- G finite abelian (with discrete topology)
  We saw earlier $G \simeq \hat{G}$.

- $G = \mathbb{R}$, $\hat{\mathbb{R}} \simeq \mathbb{R}$
  Given $y \in \mathbb{R}$, define $\chi_y \in \hat{\mathbb{R}}$ given by $\chi_y(x) = e^{2\pi i x y}$

- $G = \mathbb{R}/\mathbb{Z}$, $\widehat{\mathbb{R}/\mathbb{Z}} \simeq \mathbb{Z}$
  For $n \in \mathbb{Z}$, $\chi_n(x) = e^{2\pi i n x}$

- $G = \mathbb{Z}$, $\hat{\mathbb{Z}} \simeq \mathbb{R}/\mathbb{Z}$
  For $\alpha \in \mathbb{R}/\mathbb{Z}$, $\chi_\alpha(n) = e^{2\pi i n\alpha}$

- $G = \mathbb{R}^\times$, then $\mathbb{R}^\times \simeq \{\pm 1\} \times \mathbb{R}^\times_{>0} \simeq \{\pm 1\} \times \mathbb{R}$
  ($\mathbb{R} \simeq \mathbb{R}^\times_{>0}$ via exponential map)

Given $(\varepsilon, \sigma) \in (\mathbb{Z}/2\mathbb{Z}) \times \mathbb{R}$, define character given by $x \longmapsto \operatorname{sgn}(x)^{\varepsilon} \, |x|^{i\sigma}$.

Note: The map $\chi \mapsto \chi(g)$ gives an element of $\hat{\hat{G}}$, for all $g \in G$.

Remark: If G is locally compact, then $G \to \hat{\hat{G}}$ is an isomorphism.

Definition (Fourier transform)

- If $f \in L^1(\mathbb{R})$ (functions $f: \mathbb{R} \to \mathbb{C}$ s.t. $\int |f|\, dx < \infty$), define Fourier transform $\mathcal{F}(f)(y) = \hat{f}(y) = \int_{\mathbb{R}} f(x) e^{-2\pi i x y}\, dx$
- If $f \in L^2(\mathbb{R}/\mathbb{Z})$, $\mathcal{F}(f)(n) = \hat{f}(n) = \int_{\mathbb{R}/\mathbb{Z}} f(x) e^{-2\pi i n x}\, dx$, for $n \in \mathbb{Z}$.
- If $f: \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}$, then $\hat{f}: \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}$

$$\hat{f}(\alpha) = \frac{1}{\sqrt{q}} \sum_{a \bmod q} f(a) e^{\frac{-2\pi i \alpha a}{q}}.$$

In general, if G is a locally compact abelian group, then there is unique left translation invariant measure on G (up to scaling)

Haar measure.

- for $G = \mathbb{R}$, Haar measure is the usual Lebesgue measure
- G is discrete, then Haar measure is the counting measure "$\int f \, dg$" $= \sum_{g \in G} f(g)$.
- $G = \mathbb{R}^{\times}_{>0}$, integral given by Haar measure is $\int f(x) \frac{dx}{x}$, since $\frac{dx}{x}$ invariant under multiplication of x by a constant.

__Definition:__ If G locally compact abelian group with Haar measure dg and $f \in L^2(G)$, define $\hat{f} : \hat{G} \to \mathbb{C}$ given by

$$\hat{f}(\chi) = \int_G f(g) \overline{\chi(g)} \, dg .$$

__Definition__ (Schwartz spaces)

- let $S(\mathbb{R}/\mathbb{Z})$ be the space of all infinitely differentiable functions $f : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$
- let $S(\mathbb{R})$ be the space of all infinitely differentiable functions $f : \mathbb{R} \to \mathbb{C}$ such that for all $k, j \in \mathbb{N}$, $f^{(j)}(x) = O_{k,j}(|x|^{-k})$.

• $S(\mathbb{Z})$ the space of all functions $f: \mathbb{Z} \to \mathbb{C}$ such that $f(x) = O_k(|x|^{-k})$, for all $k \in \mathbb{N}$.

<u>Theorem</u>: • Let $f \in S(\mathbb{R})$. Then $\hat{f} \in S(\mathbb{R})$ and $f(x) = \int_{\mathbb{R}} \hat{f}(y) e^{2\pi i x y} dy = \hat{\hat{f}}(-x)$.

(Fourier inversion formula)

• Let $g \in S(\mathbb{R}/\mathbb{Z})$. Then $\hat{g} \in S(\mathbb{Z})$ and $g(x) = \sum_{n \in \mathbb{Z}} \hat{g}(n) e^{2\pi i n x}$.

• Let $g: \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}$. Then

$$g(x) = \frac{1}{\sqrt{q}} \sum_{\alpha} \hat{g}(\alpha) e^{\frac{2\pi i x \alpha}{q}}.$$

Proof: exercise.